Maths Monthly Problems

TCDmath Problem Group

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MM-11688 Consider $f : \mathbb{N}^3 \to \mathbb{N}$ such that

$$\lim_{a\to\infty} \inf_{b,c,d\in\mathbb{N},\ b<a} (f(a,c,d) - f(b,c,d)) = \infty.$$

Show that for $B \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that

$$f(a,c,d) = k \implies \max(c,d) > B.$$

Answer: *By hypothesis, given C we can find A such that if $a \geq A$ then*

$$\inf_{b,c,d\in\mathbb{N},\ b<a} (f(a,c,d) - f(b,c,d)) \geq C$$

ie

$$f(a,c,d) \geq f(b,c,d) + C$$

for all b, c, d with $b < a$. In particular, the result holds with $A + 1$ in place of A, and A in place of b:

$$f(A+1,c,d) \geq f(A,c,d) + C.$$

Similarly,

$$f(A+2,c,d) \geq f(A+1,c,d) + C,$$

and so on. Hence

$$f(A+n,c,d) \geq f(A,c,d) + nC$$

for all $n \in \mathbb{N}$*. In particular,*

$$f(A+n, c, d) \geq nC$$

for all $n, c, d \in \mathbb{N}$*.*

Now suppose $B \in \mathbb{N}$ *is given. Let*

$$\Delta = \{(c, d) : c, d \leq B\}.$$

Note that

$$\max(c, d) > B \equiv (c, d) \notin \Delta.$$

We have to show that we can find $k \in \mathbb{N}$ *such that*

$$f(a, c, d) = k \implies (c, d) \notin \Delta.$$

There are $(B+1)^2$ *integer points in* Δ*. Choose*

$$C = 2(B+1)^2;$$

and let

$$K = NC.$$

¿From above,

$$f(a, c, d) = k \in [0, K] \implies a \leq A + N.$$

The number of triples (a, c, d) *with* $a \leq A+N$ *and* $(c, d) \in \Delta$ *is*

$$(A+N+1)(B+1)^2;$$

while the number of $k \in [0, K]$ *is*

$$NC + 1 = 2N(B+1)^2 + 1.$$

Thus if

$$2N(B+1)^2 \geq (A+N+1)(B+1)^2,$$

ie

$$N \geq A + 1,$$

then there must be some $k \in [0, K]$ *such that no triple* (a, c, d) *with* $(c, d) \in \Delta$ *satisfies*

$$f(a, c, d) = k.$$