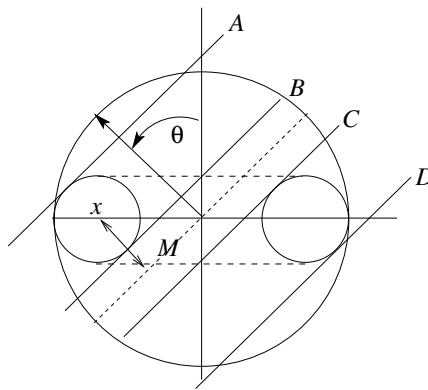


AMM problems January 2013, due before 31 May 2013

TCDmath problem group
Mathematics, Trinity College, Dublin 2, Ireland*

June 1, 2013

11687. *Proposed by Steven Finch.* Let T be a solid torus in \mathbb{R}^3 with center at the origin, tube radius 1 and spine radius r with $r \geq 1$ (so that T has volume $\pi \cdot 2\pi r$). Let P be a ‘random’ nearby plane. Find the conditional probability that, given that P meets T , that the intersection is simply connected. For what value of r is this probability maximal? (The plane is chosen by first picking a distance from the origin uniformly between 0 and $1 + r$, and then picking a normal vector independently and uniformly on the unit sphere.)



Solution (*by the TCDmath problem group.*) Choose x, y, z coordinates so that the spine circle is in the xy -plane with its centre at O . If we use (ρ, θ, ϕ) spherical coordinates, then for any ϕ_0 , the halfplane $\phi = \phi_0$ intersects T in a disc centred at $(r, \pi/2, \phi_0)$. Let S be the sphere of radius $1 + r$ around O : in spherical coordinates, the equation is $\rho = 1 + r$.

The (unconditional) sample space consists of all planes which intersect S . Planes through O are problematical, but they have probability zero, so we can ignore them and parametrise planes by the point closest to O . Then the sample space can be identified with points inside S excluding O . If we ignore vertical planes, which have probability zero, we can orient the planes so their normal vector is positive with respect to the z -direction. Every such plane has normal vector $(1, \theta, \phi)$ where $0 \leq \theta < \pi/2$.

*This group involves students and staff of the Department of Mathematics, Trinity College, Dublin. Please address correspondence either to Timothy Murphy (tim@maths.tcd.ie), or Colm Ó Dúnlaing (odunlain@maths.tcd.ie).

The torus T is rotationally symmetric around the z -axis, so the probability estimate can be reduced to two dimensions by intersecting T with any plane through the z -axis: the yz -plane, say.

Fix θ and imagine a plane with normal $(\theta, 0)$ ($\phi = 0$), floating downwards from its highest point $(1 + r, \theta, 0)$ to its lowest $(1 + r, -\theta, 0)$.

There are four critical points where it touches T , at three or four critical values (three with probability 0). List them as A, B, C, D in order of occurrence: $B = C$ with probability 0. Let M be the plane through O with normal $(\theta, 0)$, as illustrated. Then A, B, C , and D are the heights of the critical points above M . Between A and B and C and D , the intersection with T is simply connected. Between B and C , the intersection is disconnected or non-simply connected depending on θ . Hence the probability of simple connectedness is

$$\frac{A - B + C - D}{A - D} = 1 - \frac{|B - C|}{A - D}.$$

The centre of the left-hand circle is marked x on the diagram. Its coordinates are $(r, 0, 0)$. Its height above M is $r \sin \theta$, and

$$A = -D = r \sin \theta + 1, \quad B = -C = r \sin \theta - 1.$$

Therefore the probability of simple connectedness is

$$\left| \frac{r \sin \theta - 1}{r \sin \theta + 1} \right|$$

Assuming $r > 1$,

$$\int \frac{r \sin \theta - 1}{r \sin \theta + 1} d\theta = \theta + \frac{4}{\sqrt{r^2 - 1}} \tanh^{-1} \left(\frac{r + \tan(\theta/2)}{\sqrt{r^2 - 1}} \right) = F(\theta), \text{ say.}$$

(This comes from the Wolfram online calculator, adjusted to remove imaginary quantities. It can be ‘verified by differentiation.’)

Integrating the absolute value from $\theta = 0$ to $\pi/2$ involves

$$\int_0^{\sin^{-1}(1/r)} \frac{1 - r \sin \theta}{1 + r \sin \theta} d\theta + \int_{\sin^{-1}(1/r)}^{\pi/2} \frac{r \sin \theta - 1}{r \sin \theta + 1} d\theta = F(\pi/2) - 2F(\sin^{-1}(1/r)) + F(0).$$

and the probability is this quantity divided by $\pi/2$.

To simplify this expression, note that $\tan(\theta/2) = \csc \theta - \cot \theta$, with various substitutions when $\sin \theta = 1/r$. We arrive at

$$\frac{2}{\pi} \left(\frac{\pi}{2} - 2 \sin^{-1} \left(\frac{1}{r} \right) + \frac{4}{\sqrt{r^2 - 1}} \left(\tanh^{-1} \left(\frac{r + 1}{\sqrt{r^2 - 1}} \right) + \tanh^{-1} \left(\frac{r}{\sqrt{r^2 - 1}} \right) - 2 \tanh^{-1} \left(\frac{2r}{\sqrt{r^2 - 1}} - 1 \right) \right) \right).$$

This is the conditional probability that a random plane will intersect T in a simply-connected set. To maximise the probability, differentiate with respect to r . Let

$$f(\theta) = \frac{r \sin \theta - 1}{r \sin \theta + 1}, \text{ Prob} = [-F(\theta)]_0^{\sin^{-1}(1/r)} + [F(\theta)]_{\sin^{-1}(1/r)}^{\pi/2}.$$

$$\begin{aligned}\frac{df}{dr} &= \frac{2 \sin \theta}{(r \sin \theta + 1)^2} \\ \int \frac{df}{dr} d\theta &= \int \frac{2 \sin \theta}{(r \sin \theta + 1)^2} d\theta = \\ &= \frac{2 \left(\sqrt{1-r^2} \cos(\theta) + 2r(r \sin \theta + 1) \tan^{-1} \left(\frac{r + \tan(\theta/2)}{\sqrt{1-r^2}} \right) \right)}{(1-r^2)^{3/2}(r \sin(\theta) + 1)}.\end{aligned}$$

Again, this comes from Wolfram online calculator and needs to be recast since $\sqrt{1-r^2}$ is imaginary. We get

$$\int \frac{df}{dr} d\theta = \frac{2 \left(\sqrt{r^2-1} \cos \theta - 2r(r \sin \theta + 1) \tanh^{-1} \left(\frac{r + \tan(\theta/2)}{\sqrt{r^2-1}} \right) \right)}{(r^2-1)^{3/2}(r \sin \theta + 1)} = G(\theta),$$

say. The differentiated probability is

$$[-G(\theta)]_0^{\sin^{-1}(1/r)} + [G(\theta)]_{\sin^{-1}(1/r)}^{\pi/2}.$$