Solution to AMM problem 11674

TCDmath problem group
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11674 *Proposed by Pál Péter Dályay.* Let a and b be real numbers with $a < 0 < b$. Let S be the set of continuous functions f from $[0, 1]$ to $[a, b]$ with $\int_0^1 f(x)dx = 0$. Let g be a strictly increasing function from $[0, 1]$ to $\mathbb{R}$. Define ϕ from S to $\mathbb{R}$ by $\phi(f) = \int_0^1 f(x)g(x)dx$.
(a) Find $\sup_{f\in S} \phi(f)$ in terms of $a, b,$ and g.
(b) Prove that this supremum is not attained.

Answer. *From the TCDmath problem group.* Let $Y = b/(b - a)$, let $h(x)$ be the step-function

$$x \mapsto \begin{cases} a & \text{if } x < Y \\ b & \text{if } x \geq Y. \end{cases}$$

and let $B = \int_0^1 h(x)g(x)dx$.

We can assume that $g(Y) = 0$, because, if a constant is added to g, and $f \in S$, then $\int_0^1 f(x)g(x)dx$ is unchanged. If $g(Y) \neq 0$, subtract $g(Y)$ from g, and the assumption is satisfied.

The function h is discontinuous. For any continuous function $f : [0, 1] \to [a, b]$, either $f(x) > a$ on a nonempty open subset of $(0, Y)$, or $f(x) < b$ on a nonempty open subset of $(Y, 1)$: in either case, $\int_0^1 f(x)g(x)dx < \int_0^1 h(x)g(x)dx = B$. Therefore the supremum is $\leq B$, and the value B is not attained by any f in S.

For any $1 > \epsilon > 0$, let $\delta = Y\epsilon$, let $\zeta = (1-Y)\epsilon$, and let $h^\epsilon : [0, 1] \to [a, b]$ be the piecewise-linear function

$$x \mapsto \begin{cases} a & \text{if } 0 \leq x \leq Y - \delta \\ a\left(1 - \frac{x-(Y-\delta)}{\delta}\right) & \text{if } \quad Y - \delta \leq x \leq Y \\ b\left(\frac{x-Y}{\zeta}\right) & \text{if } \quad Y \leq x \leq Y + \zeta \\ b & \text{if } \quad Y + \zeta \leq x \leq 1. \end{cases}$$

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Integrating h^ϵ:

$$a(Y-\delta)+(a/2)\delta+(b/2)\zeta+b(1-Y-\zeta)=$$
$$aY(1-\epsilon)+\frac{aY\epsilon}{2}+\frac{b(1-Y)\epsilon}{2}+b(1-Y)(1-\epsilon)=$$
$$(1-\epsilon)\left(\frac{ab-ba}{b-a}\right)+\frac{\epsilon}{2}\left(\frac{ab-ba}{b-a}\right)=0$$

and

$$B-\int_0^1 h^\epsilon(x)g(x)dx=\int_{Y-\delta}^{Y}(a-h^\epsilon(x))g(x)dx+\int_{Y}^{Y+\zeta}(b-h^\epsilon(x))g(x)dx$$
$$\leq \delta(b-a)|g(Y-\delta)|+\zeta(b-a)|g(Y+\zeta)|=-\epsilon bg(Y-\delta)-\epsilon ag(Y+\zeta)\leq$$
$$(b|g(0)|+|a|g(1))\epsilon \quad \text{is } O(\epsilon).$$

Therefore

$$\sup_{f\in S}\phi(f)=B=a\int_0^{b/(b-a)}g(x)dx+b\int_{b/(b-a)}^{1}g(x)dx,$$

and the supremum is not attained by any f in S. ∎