

Answers to AMM Problems 11658, 11659, 11660, 11661

TCDmath problem group
Mathematics, Trinity College, Dublin 2, Ireland*

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MM-11658 Let V be the vector space over \mathbb{R} of all (countably infinite) sequences (x_1, x_2, \dots) of real numbers, equipped with the usual addition and scalar multiplication. For $v \in V$, say that v is *binary* if $v_k \in \{0, 1\}$ for $k \geq 1$, and let B be the set of all binary members of V . Prove that there exists a subset I of B with cardinality 2^{\aleph_0} that is linearly independent over \mathbb{R} . (An infinite subset of a vector space is linearly independent if all its finite subsets are linearly independent).

Answer: (Solved by TCDmath problem group.) Let q_1, q_2, \dots be an enumeration of the set \mathbb{Q} of rational numbers. For each real number a , let b_a be the sequence

$$b_{a,i} = \begin{cases} 1 & \text{if } a \leq q_i \\ 0 & \text{otherwise.} \end{cases}$$

Let $I = \{b_a : a \in \mathbb{R}\}$. If $a_1 < a_2 \in \mathbb{R}$ then $b_{a_1} \neq b_{a_2}$ since there exists a rational q with $a_1 < q < a_2$. Therefore $|I| = |\mathbb{R}|$. Also, $|\mathbb{R}| = 2^{\aleph_0}$ by the Schröder-Bernstein Theorem, so it is enough to show that the binary sequences b_a are linearly independent over \mathbb{R} .

Otherwise there exists a relation

$$\alpha_1 b_{a_1} + \dots + \alpha_k b_{a_k} = \vec{0}$$

where all the coefficients $\alpha_j \in \mathbb{R}$ are nonzero. We may assume that $a_1 < a_2 < \dots < a_k$. There exists a rational number q_i such that

$$a_1 \leq q_i < a_2.$$

But $b_{a_1,i} = 1$ and $b_{a_j,i} = 0$ for $j > 1$, hence $\alpha_1 = 0$, a contradiction. ■

MM-11659 Let x be real with $0 < x < 1$, and consider the sequence $\langle a_n \rangle$ given by $a_0 = 0$, $a_1 = 1$, and for $n > 1$,

$$a_n = \frac{a_{n-1}^2}{xa_{n-2} + (1-x)a_{n-1}}.$$

*This group involves students and staff of the Department of Mathematics, Trinity College, Dublin. Please address correspondence either to Timothy Murphy (tim@maths.tcd.ie), or Colm Ó Dúnlain (odunlain@maths.tcd.ie).

Show that

$$\lim_{n \rightarrow \infty} \frac{1}{a_n} = \sum_{k=-\infty}^{\infty} (-1)^k x^{k(3k-1)/2}.$$

Answer: (Solved by TCDmath problem group.)

For $n > 0$, the numbers a_n are clearly positive, and

$$\frac{a_{n-1}}{a_n} = x \frac{a_{n-2}}{a_{n-1}} + 1 - x.$$

Let $y_n = a_{n-1}/a_n$, $n \geq 1$, so $y_1 = 0$ and for $n > 1$,

$$y_n - xy_{n-1} = 1 - x.$$

Treating x as a constant, this is a linear recurrence in y_n . The constant $y_n = 1$ is a particular solution, and x^{n-1} satisfies the homogeneous recurrence, and $y_1 = 0$, so the solution is

$$y_n = 1 - x^{n-1}.$$

$$\frac{1}{a_n} = \frac{a_1}{a_n} = \prod_{k=2}^n \frac{a_{k-1}}{a_k} = \prod_{k=2}^n y_k = \prod_{k=2}^n (1 - x^{k-1}) = \prod_{k=1}^{n-1} (1 - x^k).$$

In the limit, as $n \rightarrow \infty$, this becomes the infinite product

$$\phi(x) = \prod_{k=1}^{\infty} (1 - x^k)$$

which is Euler's function. The Euler identity or pentagonal number theorem (see Wikipedia) says the latter equals

$$\sum_{k=-\infty}^{\infty} (-1)^k x^{3k(3k-1)/2}$$

as required. ■

MM-11660 Consider the following differential equation:

$$s''(t) = -s(t) - s(t)^2 \operatorname{sgn}(s'(t)),$$

where $\operatorname{sgn}(u)$ denotes the sign of u . Show that if $s(0) = a$ and $s'(0) = b$ with $ab \neq 0$, then (s, s') tends to $(0, 0)$ with $\sqrt{s^2 + s'^2} \leq C/t$ as $t \rightarrow \infty$, for some $C > 0$.

Answer: (Solved by TCDmath problem group.) The problem as stated is slightly inaccurate. There are solutions $s(t)$ (unique up to time translation $t \rightarrow c + t$, which is always possible for an autonomous differential equation such as this) tending to ± 1 .

Let us start by giving the solution $s(t)$ decreasing monotonely to 1. (The solution tending to -1 is $-s(t)$.) We have $s'(t) < 0$, so the equation reads

$$s'' = -s + s^2.$$

If we set

$$s = 1 + S$$

then the equation becomes

$$S'' = S + S^2.$$

Let

$$X(t) = \frac{1}{\sinh^2 t}.$$

Then

$$X' = -2 \frac{\cosh t}{\sinh^3 t},$$

and

$$\begin{aligned} X'' &= -2 \frac{1}{\sinh^2 t} + 6 \frac{\cosh^2 t}{\sinh^4 t} \\ &= \frac{-2 \sinh^2 t + 6 \cosh^2 t}{\sinh^4 t} \\ &= \frac{4 \sinh^2 t + 6}{\sinh^4 t} \\ &= 4S + 6S^2. \end{aligned}$$

Setting $Y = 2X/3$,

$$Y'' = 4Y + 4Y^2,$$

and finally setting

$$S(t) = Y(t/2) = \frac{2}{3} \frac{1}{\sinh^2(t/2)}$$

we have

$$S'' = S + S^2.$$

To turn to the problem, as stated, let Let

$$E(t) = s'(t)^2 + s(t)^2.$$

(We will call this the energy.) Then

$$\begin{aligned} \frac{dE}{dt} &= 2(s'' + s)s' \\ &= -2 \operatorname{sgn}(s') s' s^2 \\ &= -2|s'|s^2. \end{aligned}$$

Thus $E(t)$ is decreasing and positive, and so converges, say

$$E(t) \rightarrow c^2$$

for some $c \geq 0$.

There are two possibilities:

- (A) Either $s(t)$ is monotone for sufficiently large t ; or
 (B) $s'(t)$ changes sign infinitely often, for arbitrarily large t .

We show first that case (A) leads to the two solutions above. We may suppose that $s(t)$ is decreasing for large t , ie $s'(t) < 0$; for if $s(t)$ is a solution of the given equation with $s(t)$ decreasing then $-s(t)$ is a solution with $s(t)$ increasing,

Since $s(t)$ is decreasing and bounded, it is convergent, say

$$s(t) \rightarrow \ell \text{ as } t \rightarrow \infty.$$

Since $E(t)$ is also convergent, so also is $s'(t)$; and evidently $s'(t) \rightarrow 0$. It follows (eg by the Mean Value Theorem) that $s''(t)$ must take arbitrarily small values for certain arbitrarily large values of t . Hence from the given equation

$$s'' = -s(1 - s)$$

it follows that

$$\ell = c = 0 \text{ or } 1.$$

Suppose first that $c = 0$. Then

$$s'' = -s + s^2,$$

with $s > 0, s \rightarrow 0, s' < 0, s' \rightarrow 0$. Multiplying by s' and integrating,

$$E = s'^2 + s^2 = C + \frac{2}{3}s^3.$$

Since $s \rightarrow 0$ and $s' \rightarrow 0$ as $t \rightarrow \infty$, it follows that $C = 0$. Thus

$$s'^2 = s^2 \left(\frac{2}{3}s - 1 \right),$$

ie

$$s' = -s \sqrt{\frac{2}{3}s - 1}.$$

We can solve this explicitly. We may suppose that $0 < s < 3/2$. Let

$$s = \frac{3}{2} \frac{1}{\sin^2 \theta},$$

where $0 < \theta < \pi/2$. Then

$$s' = -\frac{3 \cos \theta}{\sin^3 \theta} \theta',$$

while

$$\frac{2}{3}s - 1 = \frac{1}{\sin^2 \theta} - 1 = \frac{\cos^2 \theta}{\sin^2 \theta},$$

and so

$$\sqrt{\frac{2}{3}s - 1} = \frac{\cos \theta}{\sin \theta},$$

and

$$s\sqrt{\frac{2}{3}s - 1} = \frac{\cos \theta}{\sin^3 \theta},$$

Thus

$$3\theta' = 1,$$

and so

$$\theta = C + t/3.$$

Hence

$$s = \frac{3}{2} \frac{1}{\sin^2(C + t/3)}.$$

Since this goes rapidly to ∞ , we conclude that there is no solution with $s(t)$ decreasing to 0.

Suppose $c = 1$. Setting $s = 1 + S$,

$$S'' = S + S^2,$$

with $S > 0, S \rightarrow 0, S' < 0, S' \rightarrow 0$.

Multiplying by S' and integrating,

$$S'^2 = S^2 + \frac{2}{3}S^3.$$

(The constant of integration is 0 since $S \rightarrow 0$ and $S' \rightarrow 0$.) Thus

$$S' = -S\sqrt{1 + \frac{2}{3}S},$$

ie

$$\frac{dS}{S\sqrt{1 + \frac{2}{3}S}} = -dt,$$

which has the explicit solution we found:

$$S(t) = \frac{2}{3} \frac{1}{\sinh^2(t/2)}.$$

Turning to case (B), where $s'(t)$ changes sign for arbitrarily large t . Suppose a local maximum M occurs at time T , and the subsequent local minimum m at time t . Then $M > 0$, or the energy would be increasing just before T ; and $m < 0$, or the energy would be increasing just after t . Thus s is oscillating between positive and negative values in a quasi-periodic manner.

Suppose local maxima $M(0), M(1), M(2), \dots$ and local minima $m(0), m(1), m(2), \dots$ occur at times $T(i)$ and $t(i)$. Let the energy at time $T(n)$, ie at the beginning of the n th cycle, be $E(n)$.

Evidently s vanishes at a unique time $Z(n)$ between $T(n)$ and $t(n)$. In the time interval $[T(n), Z(n)]$ the energy decreases by

$$\begin{aligned}\delta E &= \left[\frac{2}{3} s^3 \right]_0^{M(n)} \\ &= \frac{2}{3} M(n)^3 \\ &= \frac{2}{3} E(n)^{3/2}.\end{aligned}$$

By the same argument, the energy decrease in each of the other quarter-cycles between $T(n)$ and $T(n+1)$ lies in the range

$$\frac{2}{3} E(n+1)^{3/2} \leq \delta E \leq \frac{2}{3} E(n)^{3/2}.$$

Hence

$$\frac{8}{3} E(n+1)^{3/2} \leq E(n) - E(n+1) \leq \frac{8}{3} E(n)^{3/2}.$$

This shows incidentally that

$$c = \lim_{t \rightarrow \infty} E(t) = 0,$$

and so

$$(s(t), s'(t)) \rightarrow 0 \text{ as } t \rightarrow \infty.$$

If

$$F(x) = \frac{C^2}{(c+n)^2}$$

then

$$F(n) - F(n+1) = \frac{2C^2}{(c+n+\theta)^3},$$

where $0 < \theta < 1$. Thus

$$\frac{2C^2}{(c+n+1)^3} < F(n) - F(n+1) < \frac{2C^2}{(c+n)^3},$$

ie

$$\frac{2}{C}F(n+1)^{3/2} < F(n) - F(n+1) < \frac{2}{C}F(n)^{3/2}.$$

Setting $C = \frac{2}{3}$

$$3F(n+1)^{3/2} < F(n) - F(n+1) < 3F(n)^{3/2}.$$

We show by induction that if we start with sufficiently small $E(0)$, and choose c so that

$$F(0) \geq E(0),$$

ie

$$\frac{8}{3c^2} > E(0),$$

then

$$F(r) \geq E(r)$$

for all r . Suppose this is true for $r = n$. Then

$$\begin{aligned} F(n+1) &> F(n) - 3F(n)^{3/2} \\ &\geq E(n) - 3E(n)^{3/2} \end{aligned}$$

since $f(x) = x - 3x^{3/2}$ is increasing for $x < 4/9$. Also

$$\frac{1}{3}E(n)^{3/2} < E(n+1)^{3/2}$$

if $E(0)$ is sufficiently small, since

$$\frac{E(n+1)}{E(n)} < 1 - E(n)^{1/2} \leq 1 - E(0)^{1/2}.$$

Hence

$$\begin{aligned} E(n) - 3E(n)^{3/2} &\geq E(n) - \frac{8}{3}E(n+1)^{3/2} \\ &\geq E(n+1). \end{aligned}$$

It follows that

$$E(n) = O(n^{-2})$$

It only remains to determine how long the n th cycle lasts. Since we can regard the motion as a small perturbation of simple harmonic motion

$$s'' = -s$$

with period 2π , we may expect a cycle to take approximately the same time. However, it is sufficient for our purposes to show that

$$T(n+1) - T(n) > C$$

for some $C > 0$ independent of n .

During the n th cycle,

$$E \leq E(n), \quad s \leq \sqrt{E(n)}, \quad s' \leq \sqrt{E(n)}.$$

Thus

$$-\frac{dE}{dt} = s^2 |s'| \leq E(n)^{3/2}.$$

During the first quarter of the cycle, E decreases by

$$\left[\frac{2}{3}s^3 \right]_0^{\sqrt{E(n)}} = \frac{2}{3}E(n)^{3/2}.$$

It follows that this part of the cycle must take time $\geq \frac{2}{3}$. So n cycles take time

$$t \geq \frac{2n}{3}.$$

Hence

$$n = O(t),$$

and so

$$E(n) = O(t^{-2}),$$

and

$$\sqrt{E(n)} = O(t^{-1}).$$

MM-11661 Find every function f on \mathbb{R}^+ that satisfies the functional equation

$$(1-z)f(x) = f\left(\frac{1-z}{z}f(xz)\right)$$

for $x > 0$ and $0 < z < 1$.

Answer: (Solved by TCDmath problem group.) Clearly $f(x) = x$ is a solution. We shall show that it is the only solution. We suppose in what follows that $f(x)$ is another solution.

It is convenient to bring the functional equation to additive form

$$A + F(X + B) = B + F(A + F(X)),$$

by setting

$$F(X) = \ln f(e^X), \quad f(x) = e^{F(\ln x)}$$

(and writing $X + B$ for X). Here

$$A = \ln(1 - z) - \ln z, \quad B = -\ln z$$

are any real numbers satisfying the condition

$$e^B = e^A + 1.$$

Lemma 1 For all $X \in \mathbb{R}$,

$$F(X) \leq X.$$

PROOF Since $A \neq B$, there will be a contradiction if

$$X + B = A + F(X).$$

But this is certainly possible if $F(X) > X$, since $B - A$ takes on all positive values; for $B \rightarrow 0$ as $A \rightarrow -\infty$, and so

$$B - A \rightarrow \infty \text{ as } A \rightarrow -\infty,$$

while

$$B - A = \ln(1 + e^{-A}) \rightarrow 0 \text{ as } A \rightarrow \infty.$$

Lemma 2 The function

$$G(X) = X - F(X)$$

is non-decreasing for $X \in \mathbb{R}$.

PROOF Applying the last Lemma to the right-hand side of the functional equation for $F(X)$,

$$A + F(X + B) \leq B + A + F(X),$$

ie

$$(X + B) - F(X + B) \geq X - F(X),$$

ie

$$G(X + B) \geq G(X)$$

for all $B > 0$.

Lemma 3 $G(X) \rightarrow 0$ as $X \rightarrow -\infty$.

PROOF Suppose

$$G(X) \rightarrow c \text{ as } X \rightarrow \infty.$$

Re-writing the functional equation,

$$F(X + B) - (F + B) = (F(A + F(X)) - (A + F(X)) + (F(X) - X),$$

ie

$$G(X + B) = G(A + F(X)) + G(X).$$

Since $F(X) \leq X$,

$$F(X) \rightarrow -\infty \text{ as } X \rightarrow -\infty.$$

for all $B > 0$. Hence

$$c = c + c,$$

ie

$$c = 0.$$

Lemma 4 $G(X) \rightarrow \infty$ as $X \rightarrow \infty$.

PROOF Suppose

$$G(X) \rightarrow C \text{ as } X \rightarrow \infty.$$

Since

$$F(X) \rightarrow \infty \text{ as } X \rightarrow \infty$$

it follows from the equation for $G(X)$ in the proof of the last Lemma that

$$C = C + C,$$

ie

$$C = 0.$$

But $G(X)$ is non-negative and increasing. Hence

$$G(X) = 0 \text{ for all } X,$$

ie

$$F(X) = X \text{ for all } X \in \mathbb{R},$$

or

$$f(x) = x \text{ for all } x \in \mathbb{R}^+.$$

Corollary *The set of points where $F(X)$ is discontinuous is enumerable.*

Lemma 5 *The function $F(X)$ is injective.*

PROOF Suppose to the contrary

$$F(X_1) = F(X_2) \text{ with } X_1 < X_2.$$

Then it follows from the functional equation that

$$F(X_1 + B) = F(X_2 + B)$$

for all $B > 0$. In other words,

$$F(X + \omega) = F(X)$$

for all $X \geq X_1$, with $\omega = X_2 - X_1$. Thus $F(X)$ is periodic for $X \geq X_1$.

$F(X)$ cannot have arbitrarily small periods; for if $a \leq X \leq b$ then

$$a - F(a) \leq X - F(X) \leq b - F(b),$$

from which it follows that if $F(a) = F(b)$ then

$$F(a) - (b - a) \leq F(X) \leq F(a) + (b - a).$$

Thus the variation of $F(X)$ cannot exceed twice the period.

It follows that $F(X)$ has a minimal period ω for $X \geq X_1$.

Writing $X - B$ for X in the fundamental equation,

$$A + F(X) = B + F(A + F(X - B)).$$

Thus

$$F(A + F(X_1 - B)) = F(A + F(X_2 - B)).$$

It follows that either

$$F(X_1 - B) = F(X_2 - B),$$

or else

$$F(X'_1) = F(X'_2),$$

where

$$X'_1 = A + F(X_1), \quad X'_2 = A + F(X_2).$$

In either case the start of the periodicity is brought back.

Moreover $A = -B$ if $A = -\ln \phi$, where $\phi = (\sqrt{5} + 1)/2$. Since $A + F(X_1) \leq A + X_1$, it follows that the periodicity can be brought back by at least $\ln \phi$; so by repetition the periodicity extends over the whole of \mathbb{R} .

But this contradicts the fact that since $F(X) \leq X$,

$$F(X) \rightarrow -\infty \text{ as } X \rightarrow -\infty.$$

Lemma 6 $F(X)$ is continuous.

PROOF Suppose $F(X)$ has a discontinuity at X_0 . Since $X - F(X)$ is increasing, it follows that

$$m = \lim_{X \rightarrow X_0^-} F(X) \text{ and } M = \lim_{X \rightarrow X_0^+} F(X)$$

are defined, with

$$m > M.$$

Let us choose A so that $F(X)$ is continuous at $A + m$, $A + M$ and $X_0 + B$. (This is possible because each choice needs only avoid an enumerable set of points.) Since $F(X)$ is continuous at $A + m$ and $A + M$,

$$\begin{aligned} F(A + F(X)) &\rightarrow F(A + m) \text{ as } X \rightarrow X_0^- \\ F(A + F(X)) &\rightarrow F(A + M) \text{ as } X \rightarrow X_0^+ . \end{aligned}$$

But by the last Lemma,

$$F(A + m) \neq F(A + M).$$

Hence $F(A + F(X))$ is discontinuous at X_0 .

It follows from the fundamental equation that $F(X)$ is discontinuous at $X + B$, contrary to hypothesis.

Lemma 7 $F(X)$ is strictly increasing for $X \in \mathbb{R}$.

PROOF Since $F(X)$ is continuous and injective, it is strictly increasing or decreasing. But $F(X) \rightarrow -\infty$ as $X \rightarrow -\infty$. Hence $F(X)$ is strictly increasing.

Lemma 8 $F(X) \rightarrow \infty$ as $X \rightarrow \infty$.

PROOF Suppose to the contrary $F(X) \rightarrow C$. Then from the fundamental equation,

$$A + C = B + F(A + C).$$

Since $A + C$ takes all values, this implies that

$$F(X) = X - B$$

for all X , which is absurd since B can take any positive value.

Lemma 9 For all X ,

$$F(F(X)) = F(X + \ln 2) - \ln 2.$$

PROOF This follows on setting

$$A = 0, B = \ln 2$$

in the functional equation.

Lemma 10 $F(X) \not\rightarrow \infty$ as $X \rightarrow \infty$.

PROOF From the last Lemma,

$$\begin{aligned} F(F(X)) &= F(X + \ln 2) - \ln 2 \\ &\geq F(X) - \ln 2. \end{aligned}$$

Since $F(X)$ takes all values, this implies that

$$F(X) \geq X - \ln 2$$

for all X . But this contradicts the fact that $X - F(X) \rightarrow \infty$.

We have exhausted all possibilities; it follows that the only solution is

$$F(X) = X \text{ for all } X \in \mathbb{R},$$

ie

$$f(x) = x \text{ for all } x \in \mathbb{R}^+.$$