AMM problems 11698 and 11703

TCDmath problem group
Mathematics, Trinity College, Dublin 2, Ireland*

August 31, 2013

11698. *Proposed by Timothy Hall.* Provide an algorithm that takes as input a positive integer n and a nonzero constant k and returns polynomials F and G in variables u and v such that when x^n is substituted for u, and $x+k/x$ is substituted for v, $F(u,v)/G(u,v)$ simplifies (disregarding removable singularities) to x. (For example, when $k=1$ and $n=3$, $F=u+v$ and $G=v^2-1$ will do.)

Answer. (*From the TCDmath problem group.*) Fix the constant k. In the ring $\mathbb{Q}[x,1/x]$, call an expression *balanced* if it has the form $p(x)+p(k/x)$ where $p \in \mathbb{Q}[x]$. The polynomials of degree $\leq n$ form a vector space P_n of dimension $n+1$, and if $p_1,\ldots,p_{n+1}$ forms a basis then

$$B_n = \{p(x)+p(k/x) : \ p \in P_n\}$$

is a vector space (of balanced expressions) with the same dimension with basis $p_j(x)+p_j(k/x)$, $1 \leq j \leq n+1$.

In particular, given $v = x+k/x$, the balanced expressions $v^j, 0 \leq j \leq n$, form a basis for B_n and any balanced expression can be expressed uniquely as a polynomial in v.

Given $n \geq 0$, the expression

$$x^n + kx^{n-2} + k^2x^{n-4} + \ldots + k^n x^{-n}$$

is balanced and therefore expressible as a polynomial $b_n(x+k/x)$.

Given $u = x^n$, $n \geq 1$, let

$$F(u,v) = u + kb_{n-2}(v), \quad \text{and } G(u,v) = b_{n-1}(v)$$
$$F(x^n, x+k/x)/G(x^n, x+k/x) =$$
$$\frac{x^n + kx^{n-2} + k^2x^{n-4} + \ldots + k^{n-1}x^{2-n}}{x^{n-1} + kx^{n-3} + k^2x^{n-5} + \ldots + k^{n-1}x^{1-n}} = x.$$

To compute F and G, the main task is to calculate the coefficients of $b_n(v)$, which can be done using Gauss-Jordan elimination. ∎

---

[*] This group involves students and staff of the Department of Mathematics, Trinity College, Dublin. Please address correspondence either to Timothy Murphy (`tim@maths.tcd.ie`), or Colm Ó Dúnlaing (`odunlain@maths.tcd.ie`).

11703. *Proposed by Richard Bagby.* For $\lambda > 0$, let $\Gamma(\lambda) = \{(x,y,z) \in \mathbb{R}^3 : z \geq \lambda\sqrt{x^2+y^2}\}$, and let $C(\lambda)$ be the (half-cone) boundary of $\Gamma(\lambda)$. Prove that every point in the interior of $\Gamma(\lambda)$ is the focus of at least one ellipse in $C(\lambda)$, and find the largest μ such that every ellipse in $C(\lambda)$ has at least one focus in $\Gamma(\mu)$.

Answer. *(From the TCDmath problem group.)* See Besant, W.H.: Geometrical Conic Sections; also `http://en.wikipedia.org/wiki/Dandelin_spheres[1].`

An *inscribed sphere* S is one centred on the positive z-axis and tangent to $C(\lambda)$ in a horizontal circle, which we denote $Z(S)$.

It is known (see [Besant]) that if a plane P is tangent to an inscribed sphere S at a point $p \in S\backslash Z(S)$, then p is a focus of the conic section $C(\lambda) \cap P$.

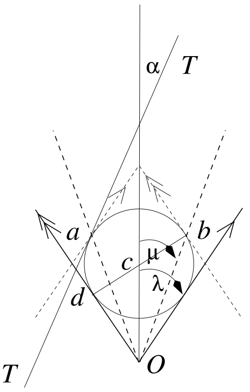
Given any point p interior to $\Gamma(\lambda)$, there is a unique inscribed sphere S such that p is on the lower component of $S\backslash Z(S)$. Let T be the plane tangent to S at p. Then $C(\lambda)$ is a conic section, contained in the truncated cone below $Z(S)$ in $C(\lambda)$, so it is bounded and therefore an ellipse. This answers the first part.

If P is a plane such that $P \cap C(\lambda)$ is an ellipse, there are two inscribed spheres touching P, one meeting it from above at the 'upper focus,' and the other from below at the 'lower focus.' From the first part every point inside $\Gamma(\lambda)$ is the upper focus of an ellipse in $C(\lambda)$. We argue that in these ellipses, the lower focus is confined to a set M which will turn out to be a narrower cone $\Gamma(\mu)$.

Fix an inscribed sphere S and let U be the upper component of $S\backslash Z(S)$. Given a plane T tangent to U, let $\alpha(T)$ be the angle T makes with the z-axis. $T \cap C(\lambda)$ is a hyperbola, parabola, or ellipse, according as $\alpha(T) < \lambda$ (respectively, $= \lambda, > \lambda$). Thus $U \cap M$ is an open disc in U bounded by the circle C, where, writing T_p for the tangent plane to s at p,

$$C = \{p \in U : \ \alpha(T_p) = \lambda\}.$$

See the illustration.



Since $\Gamma(\lambda)$ is invariant under horizontal rotation, we can assume $y = 0$, concentrating on the xz plane. Let $c = (0,0,h)$ be the centre of S. For $T_p \cap C(\lambda)$ to be an ellipse, p must be on the arc ab (a and b represent points on C). Let $\mu = \widehat{cOb}$ where the apex is at O, c is at the centre of the sphere, and the line tangent to the sphere at b is parallel to the opposite cone boundary, as illustrated.

$$
\begin{gathered}
c = (0, 0, h) \\
\|d\| = h \cos \lambda \\
d = (-h \cos \lambda \sin \lambda, 0, h \cos^2 \lambda) \\
\vec{dc} = (h \cos \lambda \sin \lambda, 0, h \sin^2 \lambda) \\
b = c + \vec{dc} = (h \cos \lambda \sin \lambda, 0, h(1 + \sin^2 \lambda)) = \|b\|(\sin \mu, 0, \cos \mu) \\
\mu = \tan^{-1} \left(\frac{\cos \lambda \sin \lambda}{1 + \sin^2 \lambda} \right).
\end{gathered}
$$

Therefore μ is the same for all inscribed spheres S and M is the interior of $\Gamma(\mu)$.